LECTURE 1

Introduction to Electrodynamics. Vector Algebra. Vector operations. The Gradient, the Divergence, the Curl.

Introduction to Electrodynamics

An electrodynamics is a division of science, which describes the process of the excitation and propagation of electromagnetic waves in different media. Electromagnetic deals with the study of electric and magnetic fields.

Electrodynamics science is associated with charged bodies in motion and varying electric and magnetic fields. By the fact that a moving charge produces a magnetic field, electrodynamics is concerned with effect such as magnetism, electromagnetic radiation, and electromagnetic induction, including such practical applications as the electric generator and electric motor.

The course of technical electrodynamics includes the study of the theory of electromagnetic processes and electrodynamic of devices technique. It covers the wide region of electromagnetic phenomena from the propagation of waves in the outer or near-Earth spaces to the processes in the electromagnetic devices.

In this book we present the classical electrodynamics, which was first systematically explained by the physicist James Clerk Maxwell.

James Clerk Maxwell (13 June 1831 – 5 November 1879) was a Scottish theoretical physicist and mathematician. His most significant achievement was the development of the classical electromagnetic theory, synthesizing all previous unrelated observations, experiments and equations of electricity, magnetism. His set of equations—Maxwell's equations—demonstrated that electricity, magnetism and even light are all manifestations of the same phenomenon: the electromagnetic field.

Maxwell’s equations are a set of differential equations. A more recent development is quantum electrodynamics, which was formulated to explain the interaction of electromagnetic radiation with matter, to which the laws of the quantum theory apply. The physicists P.A.M. Dirac, W. Heisenberg, and W. Pauli were the pioneers in the formulation of quantum electrodynamics. By the fact that the velocities of the charged particles can become comparable with the speed of light, corrections involving the theory of relativity must be made. This branch of the theory is called relativistic electrodynamics.

Electric and magnetic fields are vector quantities and their behavior is governed by Maxwell’s equations. The mathematical formulation of Maxwell’s equations requires that we first learn the basic rules pertinent mathematical manipulation involving vector quantities.

Vector Algebra

1. Vectors and vector addition
2. Unit vectors
3. Base vectors and vector components
4. Rectangular coordinates in 2-D
5. Rectangular coordinates in 3-D
6. A vector connecting two points
7. Dot product
8. Cross product
9. Triple product
10. Triple vector product

Vector operations

A magnitude and a direction must be specified for a vector quantity, in contrast to a scalar quantity which can be quantified with just a number. Any number of vector quantities of the same type (i.e., same units) can be combined by basic vector.

The vector operations are graphically described in Fig. 1.

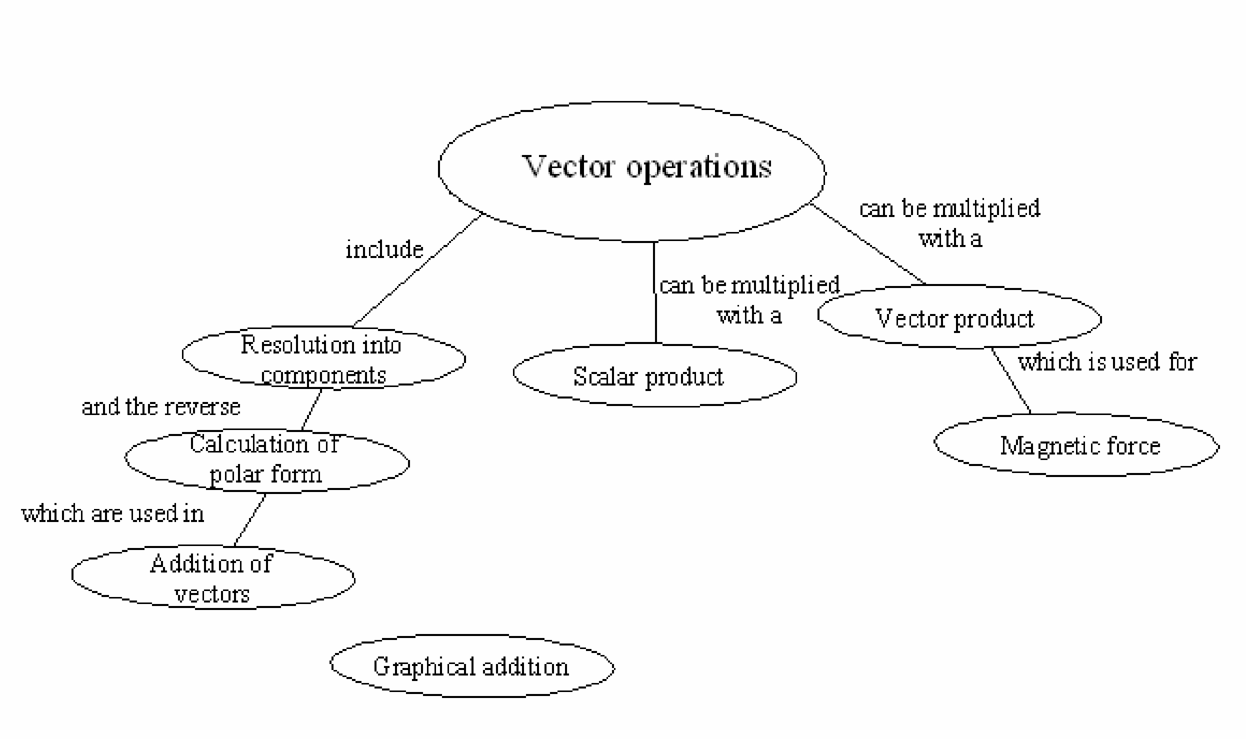

Fig. 1 The picture of vector operations

Vectors and vector addition

A scalar is a quantity that only has a magnitude (i. e. mass, temperature). However a vector is a mathematical object that has magnitude and direction. A line of given length and pointing along a given direction, such as an arrow, is the typical representation of a vector. Typical notation to designate a vector is a boldfaced character, or a character with arrow on it (i.e., **A**, $\vec{A}$). The magnitude of a vector is its length and is normally denoted by $|\mathbf{A}|$ or A.

Addition of two vectors is accomplished by laying the vectors head to tail in sequence to create a triangle such as is shown in the Fig. 2.

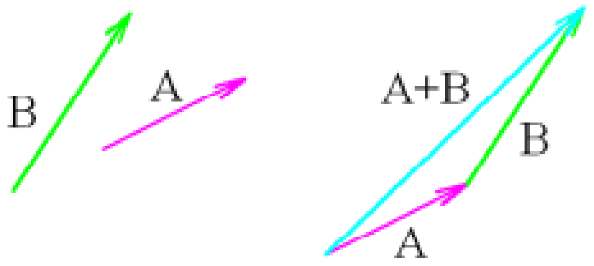

Fig. 2 The illustration of addition of two vectors

The following rules apply in vector algebra:

$$a\mathbf{P} = \mathbf{P}a$$

$$a(\mathbf{P}+\mathbf{Q}) = a\mathbf{P}+a\mathbf{Q}$$

$$\mathbf{P}+\mathbf{Q} = \mathbf{Q}+\mathbf{P}$$

where P and Q are vectors and a is a scalar.

Unit vectors

A unit vector is a vector of unit length. A unit vector is sometimes denoted by replacing the arrow on a vector with a "^" or just adding a "^" on a boldfaced character (i.e., $\hat{e}$ or $\hat{\mathbf{e}}$). Therefore,

$$|\hat{\mathbf{e}}| = 1 \tag{1}$$

Any vector can be made into a unit vector by dividing it by its length.

$$\hat{\mathbf{e}} = \frac{\mathbf{u}}{|\mathbf{u}|} \tag{2}$$

Any vector can be fully represented by providing its magnitude and a unit vector along its direction.

$$\mathbf{u} = u\hat{\mathbf{e}} \tag{3}$$

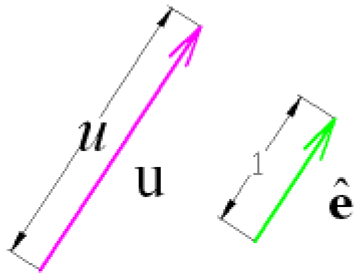

Fig. 3 Representation of any vector

Base vectors and vector components

Base vectors are a set of vectors selected as a base for representation of all the other vectors. Each vector is constructed from the addition of vectors along the base directions. For example, the vector in the Fig. 4 can be written as the sum of the three vectors $\mathbf{u_1}$, $\mathbf{u_2}$, and $\mathbf{u_3}$, each along the direction of one of the base vectors $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$, so that

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 \tag{4}$$

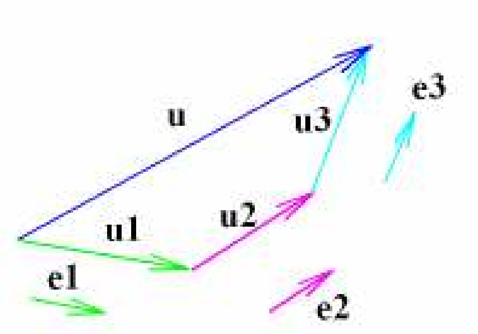

Fig. 4 The construction of any vector

Each one of the vectors $\mathbf{u_1}$, $\mathbf{u_2}$, and $\mathbf{u_3}$ is parallel to one of the base vectors and can be written as scalar multiple of that base. Let $\mathbf{u_1}$, $\mathbf{u_2}$, and $\mathbf{u_3}$ denote these scalar multipliers such that one has

$$\begin{aligned}\mathbf{u}_1 &= u_1\mathbf{e}_1\\ \mathbf{u}_2 &= u_2\mathbf{e}_2\\ \mathbf{u}_3 &= u_3\mathbf{e}_3\end{aligned} \tag{5}$$

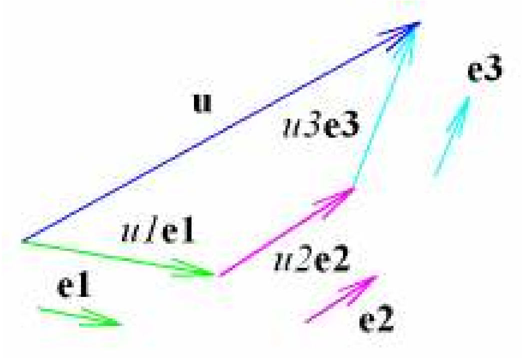

Fig. 5 The construction of any vector

The original vector u can now be written as

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 \tag{6}$$

The scalar multipliers $\mathbf{u_1}$, $\mathbf{u_2}$, and $\mathbf{u_3}$ are known as the components of **u** in the base described by the base vectors $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$.

A vector can be resolved along any two directions in a plane containing it. The Fig. 6 shows how the parallelogram rule is used to construct vectors **a** and **b** that add up to **c**.

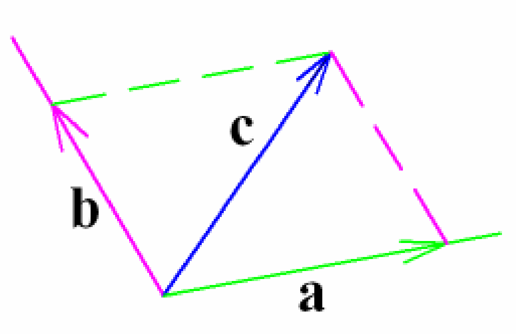

Fig. 6 The graphical illustration of the parallelogram rule

In three dimensions, a vector can be resolved along any three non-coplanar lines. The Fig. 7 shows how a vector can be resolved along the three directions by first finding a vector in the plane of two of the directions and then resolving this new vector along the two directions in the plane.

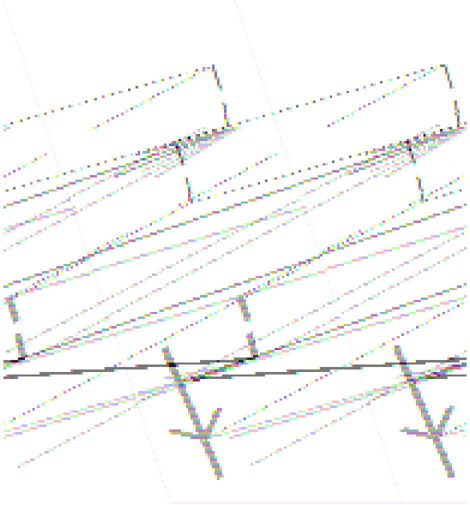

Fig. 7 Resolving of the vector in three directions

When vectors are represented in terms of base vectors and components, addition of two vectors results in the addition of the components of the vectors. Therefore, if the two vectors A and B are represented by

$$\mathbf{A} = A_1\mathbf{e_1} + A_2\mathbf{e_2} + A_3\mathbf{e_3}$$
$$\mathbf{B} = B_1\mathbf{e_1} + B_2\mathbf{e_2} + B_3\mathbf{e_3} \tag{7}$$

then,

$$\mathbf{A} + \mathbf{B} = (A_1 + B_1)\mathbf{e_1} + (A_2 + B_2)\mathbf{e_2} + (A_3 + B_3)\mathbf{e_3} \tag{8}$$

Rectangular components in 2-D

The base vectors of a rectangular Oxy coordinate system are given by the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ along the x and y directions, respectively.

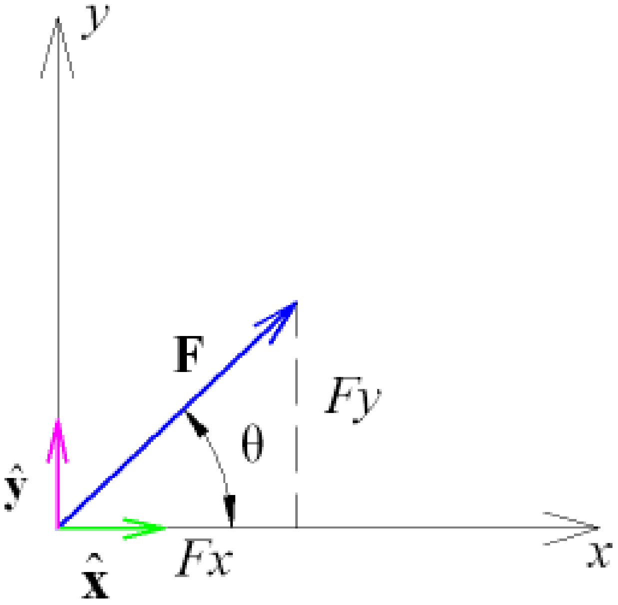

Fig. 8 The vectors in the Oxy coordinate system

Using the base vectors, one can represent any vector **F** as

$$\mathbf{F} = F_x\hat{\mathbf{x}} + F_y\hat{\mathbf{y}} \tag{9}$$

Due to the orthogonality of the bases, one has the following relations.

$$F = \sqrt{F_x^2 + F_y^2}$$
$$F_x = F\cos(\theta) \qquad (10)$$
$$F_y = F\sin(\theta)$$
$$\tan(\theta) = \frac{F_y}{F_x}$$

Rectangular coordinates in 3-D

The base vectors of a rectangular coordinate system are given by a set of three mutually orthogonal unit vectors denoted by $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ that are along the x, y, and z coordinate directions, respectively, as shown in the Fig. 9:

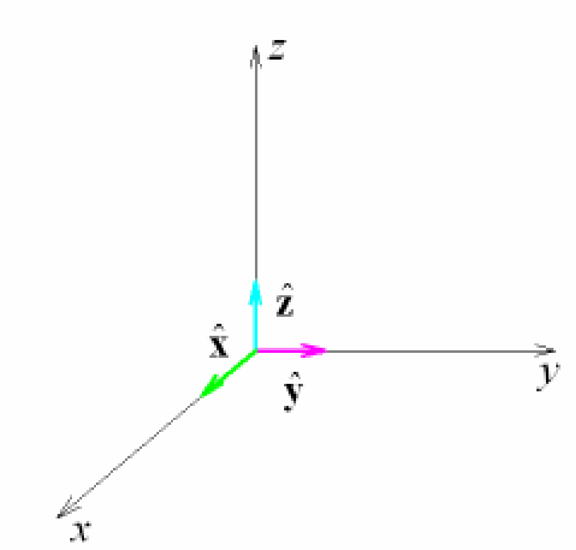

Fig. 9 The vectors in the Oxyz coordinate system

In a rectangular coordinate system the components of the vector are the projections of the vector along the x, y, and z directions. For example, in the Fig. 10 the projections of vector A along the x, y, and z directions are given by Ax, Ay, and Az, respectively.

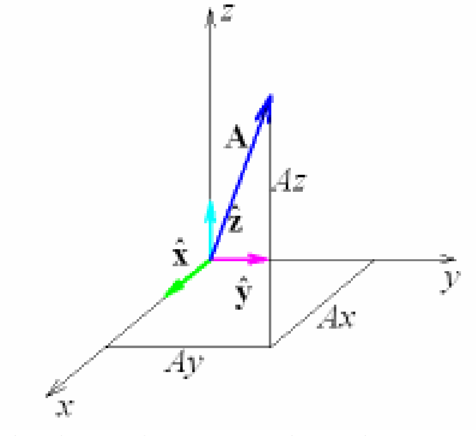

Fig. 10 The projections of vector A along the x, y, and z directions

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}} \qquad (11)$$

As a result of the Pythagorean theorem, and the orthogonality of the base vectors, the magnitude of a vector in a rectangular coordinate system can be calculated by

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2} \qquad (12)$$

Direction cosines

Direction cosines are defined as

$$l = \cos(\alpha)$$
$$m = \cos(\beta) \tag{13}$$
$$n = \cos(\gamma)$$

where the angles α, β, and γ are the angles shown in the Fig. 11. As shown in the Fig. 11, the direction cosines represent the cosines of the angles made between the vector and the three coordinate directions.

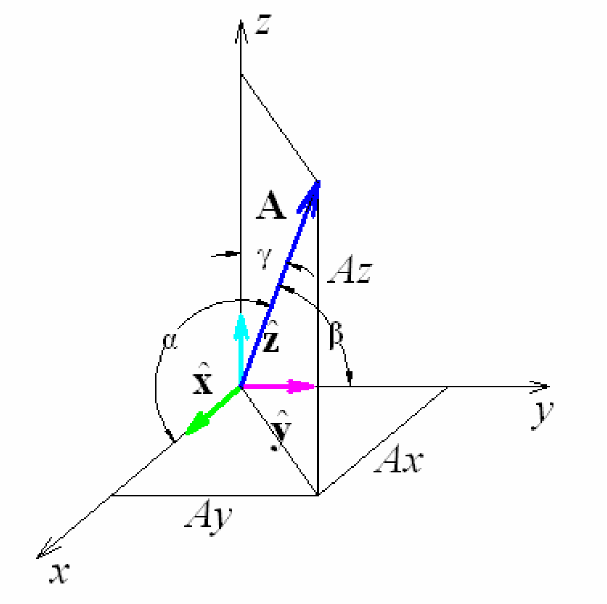

Fig. 11 Direction cosines in Oxyz coordinate system

The direction cosines can be calculated from the components of the vector and its magnitude through the relations:

$$l = \cos(\alpha) = \frac{A_x}{A},\ m = \cos(\beta) = \frac{A_y}{A},\ \cos(\gamma) = \frac{A_z}{A} \tag{14}$$

The three direction cosines are not independent and must satisfy the relation

$$l^2 + m^2 + n^2 = 1 \tag{15}$$

This results form the fact that

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) =$$
$$= \frac{A_x^2}{A^2} + \frac{A_y^2}{A^2} + \frac{A_z^2}{A^2} = 1 \tag{16}$$

A unit vector can be constructed along a vector using the direction cosines as its components along the x, y, and z directions. For example, the unit-vector $\hat{\mathbf{e}}$ along the vector **A** is obtained from

$$\hat{\mathbf{e}} = \frac{\mathbf{A}}{A} = \frac{A_x}{A}\hat{\mathbf{x}} + \frac{A_y}{A}\hat{\mathbf{y}} + \frac{A_z}{A}\hat{\mathbf{z}} =$$
$$= \cos(\alpha)\hat{\mathbf{x}} + \cos(\beta)\hat{\mathbf{y}} + \cos(\gamma)\hat{\mathbf{z}} = \tag{17}$$
$$= l\hat{\mathbf{x}} + m\hat{\mathbf{y}} + n\hat{\mathbf{z}}$$

Therefore,

$$\mathbf{A} = A\hat{\mathbf{e}} = A\cos(\alpha)\hat{\mathbf{x}} + A\cos(\beta)\hat{\mathbf{y}} + A\cos(\gamma)\hat{\mathbf{z}} \tag{18}$$

A vector connecting two points

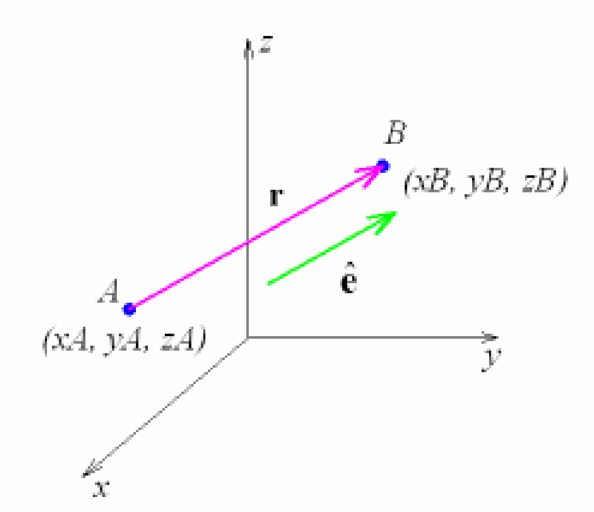

Fig. 12 The vector connecting two points

The vector connecting point A to point B (see Fig. 12) is given by

$$\mathbf{r} = (x_B - x_A)\hat{\mathbf{x}} + (y_B - y_A)\hat{\mathbf{y}} + (z_B - z_A)\hat{\mathbf{z}} \tag{19}$$

A unit vector along the line A-B can be obtained from

$$\hat{\mathbf{e}} = \frac{\mathbf{r}}{r} \tag{20}$$

A vector **F** along the line AB and of magnitude **F** can thus be obtained from the relation

$$\mathbf{F} = F\hat{\mathbf{e}} = F\frac{\mathbf{r}}{r} \tag{21}$$

Dot product

The dot product is denoted by "·" between two vectors. The dot product of vectors **A** and **B** results in a scalar given by the relation

$$\mathbf{A} \cdot \mathbf{B} = AB\cos(\theta) \tag{22}$$

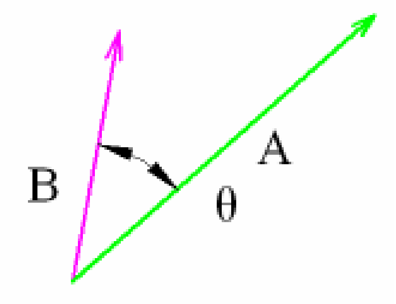

Fig. 13 The depiction of two vectors

where θ is the angle between the two vectors. Order is not important in the dot product as can be seen by the dot products definition. As a result one gets

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a} \qquad (23)$$

The dot product has the following properties.

$$\begin{aligned} a(\mathbf{b}\cdot\mathbf{c}) &= (a\mathbf{b})\cdot\mathbf{c}=\mathbf{b}\cdot(a\mathbf{c}) \\ \mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} \end{aligned} \qquad (24)$$

Since the cosine of 90^{o} is zero, the dot product of two orthogonal vectors will result in zero.

Since the angle between a vector and itself is zero, and the cosine of zero is one, the magnitude of a vector can be written in terms of the dot product using the rule

$$\mathbf{A}\cdot\mathbf{A}=A^2 \qquad (25)$$

Rectangular coordinates

When working with vectors represented in a rectangular coordinate system by the components

$$\begin{aligned} \mathbf{A} &= A_x\hat{\mathbf{x}}+A_y\hat{\mathbf{y}}+A_z\hat{\mathbf{z}} \\ \mathbf{B} &= B_x\hat{\mathbf{x}}+B_y\hat{\mathbf{y}}+B_z\hat{\mathbf{z}} \end{aligned} \qquad (26)$$

then the dot product can be evaluated from the relation

$$\mathbf{A}\cdot\mathbf{B}=A_xB_x+A_yB_y+A_zB_z \qquad (27)$$

This can be verified by direct multiplication of the vectors and noting that due to the orthogonality of the base vectors of a rectangular system one has

$$\begin{aligned} \hat{\mathbf{x}}\cdot\hat{\mathbf{y}} &= \mathbf{0} \\ \hat{\mathbf{x}}\cdot\hat{\mathbf{z}} &= \mathbf{0} \\ \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} &= \mathbf{0} \end{aligned} \qquad (28)$$

Projection of a vector onto a line

The orthogonal projection of a vector along a line is obtained by moving one end of the vector onto the line and dropping a perpendicular onto the line from the other end of the vector. The resulting segment on the line is the vector's projection.

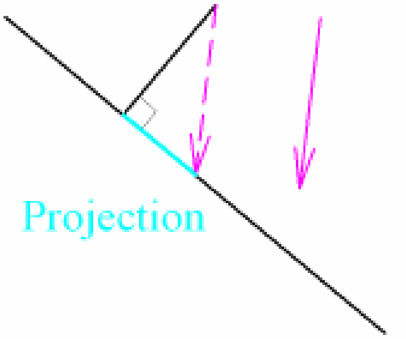

Fig. 14 The illustration of the orthogonal projection of a vector along a line

The scalar projection of vector A along the unit vector $\hat{\mathbf{e}}$ is the length of the orthogonal projection A along a line parallel to $\hat{\mathbf{e}}$, and can be evaluated using the dot product. The relation for the projection is

Scalar projection of $\mathbf{A}$ along $\hat{\mathbf{e}} = \mathbf{A}\cdot\hat{\mathbf{e}}$ (29)

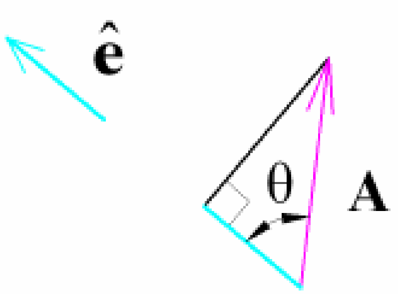

Fig. 15 The scalar projection of vector A along the unit vector $\hat{\mathbf{e}}$

The vector projection of A along the unit vector $\hat{\mathbf{e}}$ simply multiplies the scalar projection by the unit vector $\hat{\mathbf{e}}$ to get a vector along $\hat{\mathbf{e}}$. This gives the relation

Vector projection of $\mathbf{A}$ along $\hat{\mathbf{e}} = \left(\mathbf{A}\cdot\hat{\mathbf{e}}\right)\hat{\mathbf{e}}$ (30)

Rectangular coordinates. The cross product

The cross product of vectors **a** and **b** is a vector perpendicular to both a and b and has a magnitude equal to the area of the parallelogram generated from **a** and **b**. The cross product is denoted by a "×" between the vectors.

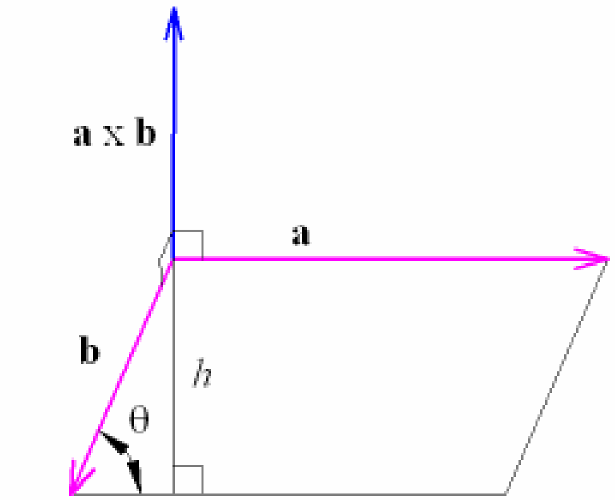

Fig. 16 The illustration of the cross product of vectors a and b

Order is important in the cross product. If the order of operations changes in a cross product the direction of the resulting vector is reversed. That is,

$$\mathbf{a}\times\mathbf{b} = \mathbf{-b}\times\mathbf{a} \tag{31}$$

The cross product has the following properties.

$$\begin{gathered} a\left(\mathbf{b}\times\mathbf{c}\right) = \left(a\mathbf{b}\right)\times\mathbf{c} = \mathbf{b}\times\left(a\mathbf{c}\right) \\ \mathbf{a}\times\left(\mathbf{b}+\mathbf{c}\right) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c} \\ \left(\mathbf{a}+\mathbf{b}\right)\times\mathbf{c} = \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c} \end{gathered} \tag{32}$$

When working in rectangular coordinate systems, the cross product of vectors a and b given by

$$\begin{gathered} \mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}} \\ \mathbf{b} = b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}} \end{gathered} \tag{33}$$

can be evaluated using the rule

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}= \qquad (34)$$

$$=\left(a_yb_z-a_zb_y\right)\hat{\mathbf{x}}\text{-}\left(a_xb_z-a_zb_x\right)\hat{\mathbf{y}}+\left(a_xb_y-a_yb_x\right)\hat{\mathbf{z}}$$

One can also use direct multiplication of the base vectors using the relations

$$\begin{aligned}\hat{\mathbf{x}}\times\hat{\mathbf{y}}&=\hat{\mathbf{z}}\hat{\mathbf{x}}\times\hat{\mathbf{x}}=\mathbf{0}\\ \hat{\mathbf{y}}\times\hat{\mathbf{z}}&=\hat{\mathbf{x}}\hat{\mathbf{y}}\times\hat{\mathbf{y}}=\mathbf{0}\\ \hat{\mathbf{z}}\times\hat{\mathbf{x}}&=\hat{\mathbf{y}}\hat{\mathbf{z}}\times\hat{\mathbf{z}}=\mathbf{0}\end{aligned} \qquad (35)$$

The triple product

The triple product of vectors a, b, and c is given by

$$\mathbf{a}\cdot\left(\mathbf{b}\times\mathbf{c}\right) \qquad (36)$$

The value of the triple product is equal to the volume of the parallelepiped constructed from the vectors. This can be seen from the Fig. 17 since

$$\begin{aligned}&Volume=abc\sin\left(\theta\right)\cos\left(\varphi\right)\\ &\mathbf{a}\cdot\left(\mathbf{b}\times\mathbf{c}\right)=a\left|\mathbf{b}\times\mathbf{c}\right|\cos\left(\varphi\right)=abc\sin\left(\theta\right)\cos\left(\varphi\right)\end{aligned} \qquad (37)$$

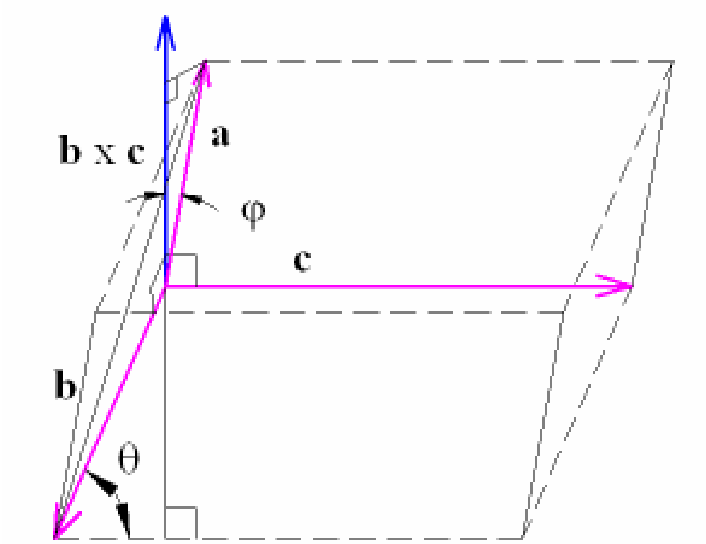

Fig. 17 The illustration of the triple product of vectors a, b, and c

The triple product has the following properties

$$\begin{aligned}&\mathbf{a}\cdot\left(\mathbf{b}\times\mathbf{c}\right)=\left(\mathbf{b}\times\mathbf{c}\right)\cdot\mathbf{a}\\ &\mathbf{a}\cdot\left(\mathbf{b}\times\mathbf{c}\right)=\mathbf{c}\cdot\left(\mathbf{a}\times\mathbf{b}\right)=\mathbf{b}\cdot\left(\mathbf{c}\times\mathbf{a}\right)\\ &\mathbf{a}\cdot\left(\mathbf{b}\times\mathbf{c}\right)=-\mathbf{a}\cdot\left(\mathbf{c}\times\mathbf{b}\right)\end{aligned} \qquad (38)$$

Rectangular coordinates

Consider vectors described in a rectangular coordinate system as

$$\begin{aligned}\mathbf{a} &= a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}} \\ \mathbf{b} &= b_x\hat{\mathbf{x}} + b_y\hat{\mathbf{y}} + b_z\hat{\mathbf{z}} \\ \mathbf{c} &= c_x\hat{\mathbf{x}} + c_y\hat{\mathbf{y}} + c_z\hat{\mathbf{z}}\end{aligned} \tag{39}$$

The triple product can be evaluated using the relation

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= \begin{vmatrix} a_x & a_y & a_z \\ a_x & a_y & a_z \\ a_x & a_y & a_z \end{vmatrix} = \\ &= \left(b_y c_z - b_z c_y\right) a_x - \left(b_x c_z - b_z c_x\right) a_y + \left(b_x c_y - b_y c_x\right) a_z\end{aligned} \tag{40}$$